

ON THE DERIVATION OF GRAD'S HYDRODYNAMIC EQUATIONS  
(COMPUTATION OF TRANSPORT COEFFICIENTS  
OF AN ARBITRARY ORDER)

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We present the method of computing transport coefficients of arbitrary order. These coefficients were introduced by Grad to describe nonequilibrium processes in a rarefied monatomic gas and we obtain them for the molecules interacting between themselves according to an exponential law, and for the rigid sphere models.

As we know, the microscopic approach to the study of a rarefied gas is based on the Boltzmann equation and the latter has no exact solution, therefore macroscopic description of the nonequilibrium phenomena in terms of the distribution function moments must suffice. The fundamental problem of the kinetic theory of gases thus reduces to the derivation of hydrodynamic equations from the kinetic equation. One of the most general methods of solution was proposed by Grad [1] and its detailed account can be found in the monographs by Klimontovich [2] or Kogan [3]. Grad expands the distribution function related to the locally Maxwellian function in the terms of tensor Hermite-Chebyshev polynomials, using a finite number of terms. In addition, his expression for the moments of the collision integral written in the terms of the coefficients of expansion of the distribution function includes the transport coefficients, which were obtained only for certain particular cases. Below we give the method of computing these quantities to an arbitrary approximation.

1. Let us consider a rarefied monatomic gas whose molecules of mass  $m$  are described by point force centers dependent on the intermolecular distance  $R$  in the following manner

$$F = Q/R^\omega$$

We exclude from our consideration all potential less steep than the Coulomb potential ( $\omega \leq 2$ ). All final formulas obtained in this paper will also be valid for the rigid sphere models, provided that we take  $\omega$  to infinity and vary  $Q$  at the same time in such a manner that

$$Q^{1/\omega-1} \rightarrow d$$

where  $d$  is the collision diameter for the rigid sphere molecules.

We shall begin our investigation by considering the following expression for the transport coefficients obtained by Grad [1]

$$\beta_{\nu\rho\tau}^{(Nrs)} \equiv \frac{1}{64\pi^3 r! s!} \int \exp\left\{-\frac{V^2 + W^2}{4}\right\} H_\rho^{(r)}\left(\frac{W-V}{2}\right) H_\tau^{(s)}\left(\frac{W+V}{2}\right) T_\nu^{(N)} d^3V d^3W$$

$$T_\nu^{(N)} \equiv \int B(0, V\sqrt{RT}) [H_\nu^{(N)}] d\theta d\psi$$

$$[H_\nu^{(N)}] \equiv H_\nu^{(N)}(\mathbf{v}') + H_\nu^{(N)}(\mathbf{w}') - H_\nu^{(N)}(\mathbf{v}) - H_\nu^{(N)}(\mathbf{w})$$

$$\mathbf{v}' = \mathbf{v} + \mathbf{h}V \cos \theta, \quad \mathbf{w}' = \mathbf{w} - \mathbf{h}V \cos \theta, \quad \mathbf{V} \equiv \mathbf{w} - \mathbf{v}, \quad \mathbf{W} \equiv \mathbf{w} + \mathbf{v}$$

Here  $H_\nu^{(N)}(\mathbf{v})$  denote the Hermite-Chebyshev tensor polynomials (see Appendix)  $\mathbf{v}$  and  $\mathbf{w}$  are the velocities of molecules before the collision,  $\mathbf{v}'$  and  $\mathbf{w}'$  - after the collision,

$V$  is the relative velocity,  $W$  is the doubled velocity of the mass centers of the colliding molecules. All velocities refer to the local coordinate system moving with the mean velocity of the gas, and are expressed in the multiples of  $\sqrt{RT}$  ( $R$  is the gas constant and  $T$  is the temperature).

Further,  $\theta$  is the angle of scattering,  $h$  is the unit vector of the bisector of  $\theta$  and  $\psi$  denotes the "azimuthal angle of collision" defining the position of the vector  $h$  in the plane perpendicular to the vector  $V$ . Range of integration in  $\psi$  is, obviously, contained between 0 and  $2\pi$ , while the limits of integration in  $\theta$  depend on the law governing the forces of interaction. For the repulsive forces,  $\theta$  varies from  $\frac{1}{2}\pi$  to  $\pi$ . If the intermolecular force is a power function of the distance between the molecules [1 and 4], then

$$B(\theta, V\sqrt{RT}) = \Lambda m (RT)^{\omega/2} v^{\omega} z \frac{dz}{d\theta} \quad \left( \Lambda \equiv \frac{1}{m} \left( \frac{2Q}{m} \right)^{2/(\omega-1)} \quad \kappa \equiv \frac{\omega-5}{\omega-1} \right)$$

where  $m$ ,  $Q$  and  $\omega$  are molecular constants, while the function  $z(\theta)$  plays the part of a collision parameter written in a dimensionless form and given implicitly by

$$\theta(z) = \int_0^z \left[ 1 - y^2 - \frac{2}{\omega-1} \left( \frac{y}{z} \right)^{\omega-1} \right]^{-1/2} dy, \quad 1 - y_0^2 - \frac{2}{\omega-1} \left( \frac{y_0}{z} \right)^{\omega-1} = 0$$

For the assumed law of interaction between the molecules,  $B(\theta, V\sqrt{RT})$  represents the product of a function of  $\theta$  and some power of  $V$ , and in the particular case of  $\omega = 5$  (Maxwellian molecules) it becomes a function of  $\theta$  only.

In the following we shall also use the differential transport coefficients  $\tau_{\nu\rho\eta}^{(Nrs)}(\kappa)$  which by definition are related to the ordinary coefficients in the following manner:

$$\beta_{\nu\rho\eta}^{(Nrs)}(\kappa) \equiv \int_0^\infty z dz \tau_{\nu\rho\eta}^{(Nrs)}(\kappa)$$

2. When computing the transport coefficients, we can use the definition of the Hermite-Chebyshev polynomials given in (A.6) in terms of the generating function. Then we can say that

$$r! s! \tau_{\nu\rho\eta}^{(Nrs)}(\kappa)$$

is the coefficient accompanying the term

$$\frac{a_\nu^N b_\rho^r c_\eta^s}{N! r! s!} \quad (\nu, \eta, \rho = 1, 2, 3)$$

in the Taylor expansion of the generating function

$$X^{(\kappa)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv \frac{\Lambda m (RT)^{\omega/2}}{6^4 \pi^3} \exp \left\{ -\frac{a^2 + b^2 + c^2}{2} \right\} \int d^3V d^3W d\psi V^\omega \exp \left\{ -\frac{V^2 + W^2}{4} + \mathbf{b} \cdot \frac{W + V}{2} + \mathbf{c} \cdot \frac{W - V}{2} \right\} [\exp(\mathbf{a} \cdot \mathbf{V}') + \exp(\mathbf{a} \cdot \mathbf{W}') - \exp(\mathbf{a} \cdot \mathbf{V}) - \exp(\mathbf{a} \cdot \mathbf{W})]$$

Here and in the following the dot will denote the inner product of tensors.

We shall say that the three-dimensional space in which the scalar function  $X(\kappa)$  is given, is conjugate to the velocity space of the molecule, vector  $\mathbf{a}$  defines a certain direction, while the vectors  $\mathbf{b}$  and  $\mathbf{c}$  define the position of two colliding molecules in the above mentioned space. Integrating with respect to  $V$ ,  $W$  and  $\psi$  we obtain the following result which will be the basis for all subsequent computations:

$$X^{(\kappa)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{2}{\sqrt{\pi}} (4RT)^{\omega/2} \Gamma \left( \frac{\omega+3}{2} \right) \sum_{l=0}^\infty \frac{(-1/2\kappa)_l (-1)^l}{(2/2)_l l!} \frac{d^l}{d\lambda^l} \left[ \exp \left\{ \frac{(\mathbf{a} + \mathbf{b} + \mathbf{c})^2}{4} \right\} \right]$$

$$-\left(\frac{a^2 + b^2 + c^2}{2}\right) (1 - \lambda) \left. X^{(0)}(\mathbf{a}, \lambda \mathbf{b}, \lambda \mathbf{c}) \right] \Big|_{\lambda=1}, \quad (z)_n \equiv \frac{\Gamma(z+n)}{\Gamma(z)} \quad (2.1)$$

$$X^{(0)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 2\pi\lambda m [\langle \exp\{(\mathbf{k}\cdot\mathbf{b}) + (\mathbf{k}^*\cdot\mathbf{c})\} \rangle + \langle \exp\{(\mathbf{k}\cdot\mathbf{c}) + (\mathbf{k}^*\cdot\mathbf{b})\} \rangle] - \exp(\mathbf{a}\cdot\mathbf{b}) - \exp(\mathbf{a}\cdot\mathbf{c}) \quad (2.2)$$

$$\mathbf{k} \equiv a \cos \theta \mathbf{e}, \quad \mathbf{k}^* \equiv a \sin \theta \mathbf{e}^*, \quad \mathbf{e} \equiv \mathbf{e}(\theta, \varphi), \quad \mathbf{e}^* \equiv \mathbf{e}(\pi/2 - \theta, \varphi + \pi) \quad (2.3)$$

Here  $\Gamma(z)$  is the gamma function,  $\mathbf{e}(\theta, \varphi)$  denotes (conditionally) a unit vector defined by two angles  $\theta$  and  $\varphi$  counted from the direction given by  $\mathbf{a}$ ,  $\theta$  is the polar angle and  $\varphi$  is the azimuthal angle, i. e. the angle defining the position of this vector in the plane perpendicular to the vector  $\mathbf{a}$ . Brackets  $\langle \rangle$  denote the process of averaging over the azimuthal angle

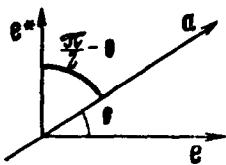


Fig. 1

$$\langle F(\varphi) \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} F(\varphi) d\varphi$$

Using (2.1) we can easily confirm that the generating function corresponding to the power model, is a linear functional of the generating function of the Maxwellian model. Its contribution into the latter can be defined as the interference of two plane

decaying waves, one of which is defined by the wave vector  $i\mathbf{a} \cos \theta \mathbf{e}$  and the other - by  $i\mathbf{a} \sin \theta \mathbf{e}^*$ , unit vectors  $\mathbf{e}$  and  $\mathbf{e}^*$  being coplanar to the vector  $\mathbf{a}$  as shown in Fig. 1.

Thus we can formally consider a collision of two Maxwellian molecules as an interference of two plane decaying waves in a conjugate space. This is true only for the Maxwellian model. We can use this approach when computing the transport coefficients.

3. Let us consider the Maxwellian case ( $\kappa = 0$ ). Using the Taylor formula we can represent differential transport coefficients as a  $(N + r + s)$ -th rank tensor derivative of the generating function, computed at the null point of the conjugate space

$$r!s!\Gamma_{\nu\epsilon\eta}^{(Nrs)} = \nabla_{\nu}^N(\mathbf{a}) \nabla_{\eta}^s(\mathbf{b}) \nabla_{\epsilon}^r(\mathbf{c}) X_{\mathbf{a}=\mathbf{b}=\mathbf{c}=\mathbf{0}} \quad (3.1)$$

When differentiating (2.2), we begin by differentiating with respect to the vectors  $\mathbf{b}$  and  $\mathbf{c}$  keeping  $\mathbf{a}$  fixed and then apply the theorem given in the Appendix (see (A.12) and (A.13)). Differentiation with respect to  $\mathbf{a}$  is performed in a straightforward manner and the resulting expression for the transport coefficient is

$$\beta_{\nu\epsilon\eta}^{(Nrs)}(0) = \frac{2\lambda m}{r!s!} \delta_{N, s+r} \sum_{n=0}^{\lfloor N/2 \rfloor} \sum_{l=l_1}^{l_2} \frac{(-1)^l [(2n)!!]^2}{(2n)!} \sigma_{lr}^{(s+r, sr)}(n, l) \times \times (\delta_{\nu\nu}^n \delta^n(\{p^l\}, \{n^{2n-l}\}) \delta_{\nu\epsilon}^{-l} \delta_{\nu\eta}^{-2n+l})_{\nu, \epsilon, \eta}^* \quad (3.2)$$

$$l_1 \equiv \max(0, 2n - s), \quad l_2 \equiv \min(2n, r)$$

where  $\min(a, b, \dots e)$  denotes the smallest and  $\max(a, b, \dots e)$  the largest member of a finite set of numbers, while the asterisk denotes the operation of symmetrization (see Appendix)

$$\sigma_{lr}^{(s+r, sr)}(n, l) \equiv \pi \int_0^{\infty} z dz I_{nl}^{(s+r, sr)}(\theta) \quad (3.3)$$

$$I_{nl}^{(s+r, sr)}(\theta) \equiv [\cos^{r+2n-l} \theta \sin^{s+l} \theta S_n^{(l-l, s-2n+l)}\left(\theta, \frac{\pi}{2} - \theta\right) +$$

$$+ \sin^{r+2n-l} \theta \cos^{s+l} \theta S_n^{(l-2n+l, r-l)} \left( \theta, \frac{\pi}{2} - \theta \right) - \delta_{no} (\delta_{r0} + \delta_{s0}) \} \quad (3.4)$$

We shall call the angular function  $J_{nl}^{(s+r, sr)}(\theta)$  the scattering indicatrix and the modulus of  $\sigma_{tr}^{Nsr}(n, l)$  — the transport scattering cross section of arbitrary order, corresponding to the Maxwellian model. Functions  $S_k^{(mn)}(\theta_1, \theta_2)$  are defined by Formula (A. 13). As an illustration we shall give several lower order transport scattering cross sections required in computing the moments of the collision integrals up to the fifth order inclusive

$$\begin{aligned} \sigma_{tr}^{(000)}(0, 0) = \sigma_{tr}^{(110)}(0, 0) = 0, \quad \sigma_{tr}^{(220)}(0, 0) = -3A, \quad \sigma_{tr}^{(211)}(0, 0) = 3A \\ \sigma_{tr}^{(220)}(1, 0) = \sigma_{tr}^{(202)}(1, 2) = \sigma_{tr}^{(211)}(1, 1) = A, \quad \sigma_{tr}^{(330)}(0, 0) = -\frac{3}{2}A, \quad \sigma_{tr}^{(312)}(0, 0) = \frac{3}{2}A \\ \sigma_{tr}^{(330)}(1, 0) = \sigma_{tr}^{(303)}(1, 2) = \sigma_{tr}^{(312)}(1, 1) = \sigma_{tr}^{(312)}(1, 2) = \frac{1}{2}A \\ \sigma_{tr}^{(404)}(0, 0) = \frac{35}{4}B - 7A, \quad \sigma_{tr}^{(404)}(1, 2) = \frac{1}{2}(A - \frac{1}{2}B) \\ \sigma_{tr}^{(404)}(2, 4) = \sigma_{tr}^{(222)}(2, 2) = \frac{3}{4}B, \quad \sigma_{tr}^{(422)}(0, 0) = \frac{35}{4}B - A, \quad \sigma_{tr}^{(422)}(1, 1) = \frac{1}{4}B \\ \sigma_{tr}^{(550)}(0, 0) = \frac{175}{8}B - 10A, \quad \sigma_{tr}^{(505)}(1, 2) = \frac{1}{8}(A - \frac{15}{4}B) \\ \sigma_{tr}^{(505)}(2, 4) = \sigma_{tr}^{(523)}(2, 2) = \sigma_{tr}^{(523)}(2, 3) = \frac{3}{8}B \\ \sigma_{tr}^{(523)}(0, 0) = \frac{1}{2}(\frac{35}{4}B - A), \quad \sigma_{tr}^{(523)}(1, 1) = \sigma_{tr}^{(523)}(1, 2) = \frac{1}{8}B \\ A = A(x) \equiv \pi \int_0^\infty dz \sin^2 \theta \cos^2 \theta, \quad B = B(x) \equiv \pi \int_0^\infty dz \sin^4 \theta \cos^4 \theta \end{aligned}$$

Here  $A$  and  $B$  represent the dimensionless effective collision cross sections. Numerical values of  $A(x)$  are given for various  $x$  in a book by Chapman and Cowling [4] where e. g.  $A(0) = 0.343$ . According to the results of [5],  $B(0) = 0.054$ . Formulas for the second and third order moments of collision integrals are given in [1, 8 and 9], and those for the fourth and fifth order moments — in [7]. It can easily be confirmed that the moments obtained with the help of the transport coefficients given in (3.2), agree with the corresponding values given in the papers quoted above.

4. Passing to the general case we note that the property of linearity of the functional (2.1) can be used to obtain the differential transport coefficients in a very simple manner. We apply the Taylor formula (3.1) and the rule of differentiation of a product. Summation over  $l$  is performed with the help of the following formula (see [10]):

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}$$

in which  $F(\alpha, \beta, \gamma; 1)$  denotes a hypergeometric series. Further, using the formula (A. 11) we have

$$\begin{aligned} \tau_{v\varphi n}^{(Nrs)}(x) = \frac{[(-1)^{N+s+r} + 1]}{2} (-1)^{1/2(N+s+r)} \frac{2^{-1/2(N+s+r)} (4RT)^{1/2} \Gamma(1/2x + 1)}{\Gamma(1/2(N+s+r+3)) r! s!} x \\ \times \sum_{i=0}^N \sum_{\substack{j=0 \\ i+j \leq N}}^r \frac{(-2)^{i+j} \Gamma(i+j+1/2(x+3)) i! j!}{\Gamma(i+j-1/2(N+s+r-x))} \times \\ \times \{ \delta_{(-)}^{1/2(N+s+r)-i-j} (v^{N-i-j}, \{\eta^{s-i}, \{p^{r-j}\} \tau_{v\varphi n}^{i+j, n}(0)\}^s, \rho, n \end{aligned}$$

where the inequality  $i + j \leq N$  indicates that summation is performed over all those

values of  $i$  and  $j$  whose sums do not exceed  $N$ .

Thus we obtain the following corollary emerging from the basic relation (2, 1): differential transport coefficients corresponding to the molecules interacting between themselves according to some power law, are linear functions of the coefficients of the Maxwell's model.

Using formulas (3, 1) and (3, 2) and the explicit form of the Kronecker delta products (see (A, 4) and (A, 5)) together with the rule of combination of ranks under the symmetrization sign (A, 1) we obtain, after the relevant interchange of the summation signs, the following final expression for the transport coefficients:

$$\beta_{\nu\mu\alpha}^{(Nrs)}(\alpha) = \frac{[(-1)^{N+s+r} + 1]}{2} (-1)^{1/2(N+s+r)} 2^{-1/2(N+r+s)+1} \frac{\Lambda n (4RT)^{1/2} \Gamma(1/2 \alpha + 1)}{\Gamma(1/2(N+r+s+3)) r! s!} \times$$

$$\times \sum_{\alpha=0}^{[1/2 N]} \sum_{\beta=0}^{[1/2 s]} \sum_{\gamma=0}^{[1/2 r]} (-1)^{\alpha+\beta+\gamma} \sigma_{tr}^{(N,r)}(\alpha, \beta, \gamma; \alpha) \{ \delta_{\nu\nu}^{\gamma} \delta_{\mu\mu}^{\beta} \delta_{\rho\rho}^{\alpha} \delta_{\nu\rho}^{\gamma} \delta_{\nu\rho}^{\beta} \delta_{\rho\nu}^{\alpha} \}^{\nu, \rho, \eta} \quad (4.1)$$

( $p, q, t \geq 0$ )

$$\sigma_{tr}^{(Nsr)}(\alpha, \beta, \gamma; \alpha) \equiv \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \sum_{l=k_1}^{l_2} \sum_{l=t_1}^{l_2} \sum_{n=n_1}^{n_2} \frac{(-2)^{i+j} \Gamma(i+j+1/2(\alpha+3))}{\Gamma(i+j+1-1/2(N+s+r-\alpha))} \times$$

$$\times \frac{[(2k)!!]^2}{(2k)!} \binom{\alpha}{k} \binom{\beta}{k+n-l} \binom{\gamma}{n} \binom{p}{i+l-2k} \binom{q}{j-l} \binom{t}{l-2n} \mathfrak{S}_{tr}^{(i+j, ij)}(k, l) \quad (4.2)$$

Here  $p, q$  and  $t$  are defined by (A, 3),  $\sigma_{tr}^{(i+j, ij)}(k, l)$  are the scattering transport cross sections corresponding to the Maxwell's model (see (3, 3) and (A, 13)) and

$$k_1 \equiv \max(0, \alpha - [1/2(N-i-j)]), \quad k_2 \equiv \min([1/2(i+j)], \alpha)$$

$$l_1 \equiv \max(0, 2k - i, k - [1/2(r-j)] + \gamma - \beta, j - q), \quad l_2 \equiv \min(2k, j, [1/2(s-i)] + k + \gamma - \beta, 2k - i + p)$$

$$n_1 \equiv \max(0, \gamma - [1/2(r-j)], [1/2(l-t+1)], l - k), \quad n_2 \equiv \min([1/2 l], \gamma, \beta + l - k)$$

For certain values of the indices  $N, r$  and  $s$ , the computed values of the coefficients coincide with those obtained directly by Grad in [1]. Moreover, we can directly confirm the following relations:

$$\beta_{\rho\eta}^{(ors)}(\alpha) = \beta_{i\rho\eta}^{(irs)}(\alpha) = \sum_{\alpha} 3_{\alpha\alpha, \rho\eta}^{(2rs)}(\alpha) = 0 \quad (r, s = 0, 1, 2 \dots)$$

which express the fact that the density, momentum and the energy of the gas are not affected by the collisions.

The fact that the scattering indicatrix of the molecules obeying a power law is a linear function of the indicatrices corresponding to the Maxwell's model, can be regarded as one of the most important results obtained here.

Using the transport coefficients of an arbitrary order obtained above, we can easily construct an infinite net of moment equations equivalent to the Boltzmann equation. To close this system, we must use one of the statistical hypotheses, e. g. the maximum probability principle [9 and 10] (see also [3]). Thus, we are now able to construct a system of exact moment equations for the the simpler molecular models, and these equations will describe an arbitrary anisotropic state of the gas in the case, when the distribution function belongs to some sufficiently wide class of functions.

All basic results obtained in this paper remain valid for mixtures of gases.

We take this opportunity to thank A. M. Obukhov for the interest and advice.

Appendix. Tensor notation used throughout this work was developed by Grad [1 and 11] and constitutes a generalization of the dyadic notation. Thus  $c_{\nu}^n$  denotes an  $n$ th order tensor monomial  $c_{\nu_1} c_{\nu_2} \dots c_{\nu_n}$ .

Here the index  $\nu$  indicates that the subscripts belong to the  $\nu$ -group. Components of the vector  $c$  in the Cartesian coordinate system are denoted by  $c_{\nu}$  ( $\nu = 1, 2, 3$ ). Similarly,  $T_{\nu\rho\sigma}^{(Nrs)}$  denotes a tensor of the  $(N + r + s)$ -th rank with  $N$  subscripts belonging to the group  $\nu$ ,  $r$  subscripts of the group  $\rho$  and  $s$  subscripts of the group  $\sigma$ :

$$\nu \sim (\nu_1, \nu_2, \dots, \nu_N), \rho \sim (\rho_1, \rho_2, \dots, \rho_r), \sigma \sim (\sigma_1, \sigma_2, \dots, \sigma_s)$$

We shall, in addition, consider the symbolic tensor monomial  $\nabla_{\nu}^n(\mathbf{a})$  which is a product of the components of the vector del operator

$$\nabla_{\nu}^n(\mathbf{a}) \equiv \nabla_{\nu_1}(\mathbf{a}) \nabla_{\nu_2}(\mathbf{a}) \dots \nabla_{\nu_n}(\mathbf{a}) \quad (\nabla(\mathbf{a}) \equiv \partial / \partial \mathbf{a})$$

where  $\nabla(\mathbf{a})$  denotes a tensor derivative with respect to  $\mathbf{a}$

Let us introduce more notations. Kronecker delta will be denoted by  $\delta_{mn}$

$$\delta_{mn} = 0 \quad (m \neq n); \quad \delta_{mn} = 1 \quad (m = n)$$

and the symbol  $\delta$  written in bold face, will denote the Kronecker delta with tensor indices

$$\delta_{ij} = 0 \quad (i \neq j); \quad \delta_{ij} = 1, \quad (i = j) \quad (i, j = 1, 2, 3)$$

Symbol  $\delta_{\nu}^n$  denotes an  $n$ th order tensor and  $p_{\nu}^n$  denotes a tensor monomial formed by taking a product of the components of the corresponding tensors. It has the indices of the given group and the order in which they appear is immaterial. For example,

$$\delta_{\nu}^n = \delta_{\nu_1 \nu_2} \delta_{\nu_3 \nu_4} \dots \delta_{\nu_{n-1} \nu_n}$$

Symmetrizing operation is performed as follows:

$$\{T_{\nu\rho\sigma}^{(Nrs)}\}_{\nu, \rho, \sigma}^* \equiv \sum T_{\nu\rho\sigma}^{(Nrs)}$$

where the sum is taken only over the permutations yielding distinct terms. Superscripts  $\nu, \rho$  and  $\sigma$  outside the curly bracket and above the asterisk, denote the group over which the symmetrizing operation is performed. In cases when only one group of indices appears or, when the choice of the group is immaterial, they can be omitted. For example,

$$\{\delta_{\nu}^2\}_{\nu} \equiv \delta_{\nu_1 \nu_2} \delta_{\nu_3 \nu_4} + \delta_{\nu_1 \nu_3} \delta_{\nu_2 \nu_4} + \delta_{\nu_1 \nu_4} \delta_{\nu_2 \nu_3} \quad \{\delta_{\nu\rho}^2\}_{\nu, \rho} \equiv \delta_{\nu_1 \rho_1} \delta_{\nu_2 \rho_2} + \delta_{\nu_1 \rho_2} \delta_{\nu_2 \rho_1}$$

Symmetrizing operation has the following properties:

$$\begin{aligned} & \left\{ \xi_{\nu}^p \xi_{\rho}^q \xi_{\sigma}^h \delta_{\nu\rho}^m \delta_{\rho\sigma}^n \delta_{\nu\sigma}^s \delta_{\nu\rho}^l p_{\nu_1 \nu_2}^l p_{\rho_1 \rho_2}^r \right\}_{\nu, \rho, \sigma}^* = \\ & = \binom{p+q}{p} \binom{m+n}{n} \binom{s+l}{s} \binom{l+r}{r} \left\{ \xi_{\nu}^{p+q} \xi_{\rho}^h \delta_{\nu\rho}^{m+n} \delta_{\nu\sigma}^{s+l} p_{\nu_1 \nu_2}^{l+r} \right\}_{\nu, \rho, \sigma}^* \quad (A.1) \end{aligned}$$

and we shall call this the rule of summation of ranks under the symmetrizing sign.

We also employ the symbolic tensor notation using the powers of the Kronecker delta, its indices belonging to several groups. The latter can be written as tensor arguments

$$\delta^{1/2(N+r+s)}(\nu_1, \nu_2, \dots, \nu_N; \rho_1, \rho_2, \dots, \rho_r; \sigma_1, \sigma_2, \dots, \sigma_s) \equiv \delta^{1/2(N+r+s)}(\{\nu^N\}, \{\rho^r\}, \{\sigma^s\})$$

where  $\{\nu^N\}$ ,  $\{\rho^r\}$  and  $\{\sigma^s\}$  denote the sets of indices, each containing, respectively,  $N$ ,  $r$  and  $s$  indices of the group  $\nu, \rho$  and  $\sigma$ . Here the symmetrization is understood to have been already performed. Using the simplest combinatorial relationships we can easily show that (see e. g. [1])

$$\delta^{1/2(N+r+s)}(\{v^N\}, \{p^r\}, \{\sigma^s\}) = \sum_{\alpha=0}^{[1/2 N]} \sum_{\beta=0}^{[1/2 s]} \sum_{\gamma=0}^{[1/2 r]} \{\delta_{v\alpha}^{\alpha} \delta_{v\beta}^{\beta} \delta_{v\gamma}^{\gamma} \delta_{v\alpha}^{\alpha} \delta_{v\beta}^{\beta} \delta_{v\gamma}^{\gamma}\}_{\alpha, \beta, \gamma}^{p, \sigma, \rho} \quad (A.2)$$

(p, q, l ≥ 0)

where

$$p \equiv \frac{N-2\alpha}{2} + \frac{s-2\beta}{2} - \frac{r-2\gamma}{2}, \quad q \equiv \frac{N-2\alpha}{2} + \frac{r-2\gamma}{2} - \frac{s-2\beta}{2}$$

$$l \equiv \frac{s-2\beta}{2} + \frac{r-2\gamma}{2} - \frac{N-2\alpha}{2} \quad (A.3)$$

and summation is performed over all positive values of p, q and l. In addition, we shall introduce tensors of the same type but with terms alternating in sign. Using uniform notation we shall write

$$\delta^{1/2(N+r+s)}(\{v^N\}, \{p^r\}, \{\sigma^s\}) \equiv \sum_{\alpha=0}^{[1/2 N]} \sum_{\beta=0}^{[1/2 s]} \sum_{\gamma=0}^{[1/2 r]} (\pm 1)^{\alpha+\beta+\gamma} \{\delta_{v\alpha}^{\alpha} \delta_{v\beta}^{\beta} \delta_{v\gamma}^{\gamma} \delta_{v\alpha}^{\alpha} \delta_{v\beta}^{\beta} \delta_{v\gamma}^{\gamma}\}_{\alpha, \beta, \gamma}^{p, \sigma, \rho} \quad (A.4)$$

(p, q, l ≥ 0)

When only two groups of indices are used, Expression (A.2) simplifies considerably, e. g.

$$\delta^n(\{p^l\}, \{\sigma^{2n-l}\}) = \sum_{\alpha=0}^{[1/2 l]} \{\delta_{p\alpha}^{\alpha} \delta_{\sigma\alpha}^{\alpha} \delta_{\sigma\alpha}^{\alpha} \delta_{\sigma\alpha}^{\alpha}\}_{\alpha}^{n, 2n-l} \quad (A.5)$$

Hermite-Chebyshev tensor polynomials were discussed by Grad in [1 and 11]. They can be obtained using the generating function F(a)

$$F(a) \equiv \exp\left\{a \cdot \xi - \frac{a^2}{2}\right\} = \sum_{n=0}^{\infty} \frac{a_{k_1} a_{k_2} \dots a_{k_n}}{n!} \cdot H_{k_1 k_2 \dots k_n}^{(n)}(\xi) \quad (A.6)$$

and possess the following property of orthonormality

$$\frac{1}{(2\pi)^{3/2}} \int \exp\left(-\frac{\xi^2}{2}\right) H_{\nu}^{(p)}(\xi) H_{\nu}^{(q)}(\xi) d^3\xi = \delta_{np} \{\delta_{\nu}^n\},$$

where the integration is performed over the whole three-dimensional space. Using the definition, we can represent the above polynomials in terms of the tensor derivative

$$H_{\nu}^{(n)}(\xi) = (-1)^n \exp(1/2 \xi^2) \nabla_{\nu}^n(\xi) \exp(-1/2 \xi^2) \quad (A.7)$$

or explicitly

$$H_{\nu}^{(n)}(\xi) = \sum_{k=0}^{[1/2 n]} (-1)^k \{\xi_{\nu}^{n-2k} \delta_{\nu}^k\} \quad (A.8)$$

The latter yields the following expansion of a tensor monomial  $\xi^{\nu}$  in terms of the Hermite-Chebyshev polynomials

$$\xi^{\nu} = \sum_{k=0}^{[1/2 n]} \{\delta_{\nu}^k H_{\nu}^{(n-2k)}(\xi)\} \quad (A.9)$$

Using Formulas (A.7) and (A.8) we easily obtain the values of the following tensor derivatives at the null point

$$\nabla_{\nu}^n(a) \exp\left(-\frac{a^2}{2}\right) \Big|_{a=0} = (-1)^n \{\delta_{\nu}^n\} \quad (A.10)$$

$$\nabla_{\nu}^N(a) \nabla_{\sigma}^s(b) \nabla_{\rho}^r(c) \exp\left\{\frac{(a+b+c)^2}{4} - \frac{a^2+b^2+c^2}{2}\right\} \Big|_{a=b=c=0} = \frac{[(-1)^{N+s+r} + 1]}{2} 2^{-1/2(N+r+s)} \delta_{(\nu)}^{1/2(N+r+s)}(\{v^N\}, \{\sigma^s\}, \{p^r\}) \quad (A.11)$$

Finally we shall formulate the theorem referred to when performing the tensor differentiation.

Let a direction 1 be given in a three-dimensional space, together with two unit vectors  $\mathbf{e} \equiv \mathbf{e}(\theta_1, \varphi)$  and  $\mathbf{e}^* \equiv \mathbf{e}(\theta_2, \varphi + \pi)$  (see Fig. 1). Then

$$\langle \mathbf{e}_\rho^r \mathbf{e}_\sigma^s \rangle = \sum_{n=0}^{[1/2 N]} \sum_{l=l_1}^{l_2} (-1)^l \frac{(2n)!}{(2n)!} \sin^l \theta_1 \sin^{2n-l} \theta_2 S_n^{(r-l, s-2n+l)}(\theta_1, \theta_2) \times \\ \times \{ \delta_{1\rho}^{r-l} \delta_{1\sigma}^{s-2n+l} \delta^n(\{\rho^l\}, \{\sigma^{2n-l}\}) \}^{\rho, \sigma} \quad (\text{A.12})$$

$$N = r + s, \quad l_1 \equiv \max(0, 2n - s), \quad l_2 \equiv \min(2n, r)$$

where the bar denotes averaging over the azimuthal angle, and functions  $S_k^{(mn)}(\theta_1, \theta_2)$  are given by

$$S_k^{(mn)}(\theta_1, \theta_2) \equiv \frac{1}{2\pi} \int_0^{2\pi} \sin^{2k} \varphi (\cos \theta_1 + i \sin \theta_1 \cos \varphi)^m (\cos \theta_2 - i \sin \theta_2 \cos \varphi)^n d\varphi \quad (\text{A.13})$$

Proof of the above theorem can be obtained in various ways, and will not be given here.

In conclusion we shall give the expansion of  $S^{(mn)}(\theta, 1/2\pi - \theta)$  in terms of the generalized spherical functions  $P_{mn}^l$ , whose properties are described in great detail in [8]

$$S_k^{(mn)}(\theta, 1/2\pi - \theta) = \frac{\Gamma(k + 1/2)}{\pi} (-i)^n \sqrt{m!n!} \sum_{l=0}^{[1/2(m+n)]} \times \\ \times \frac{\Gamma(l + 1/2) P_{1/2(m+n-2l), 1/2(m-n)}^{1/2(m+n)}(\cos 2\theta)}{(l+k)! \sqrt{(2l)! (m+n-2l)!}} \quad (\text{A.14})$$

Using the integral representation (A.13) we can easily confirm that the functions  $S_k^{(mn)}$  coincide, at  $m = 0$ , with Hegenbauer polynomials  $C_n^{-k/2}(\cos \theta)$  with the accuracy of up to the norm.

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## ON THE APPLICATION OF THE ENSKOG METHOD TO THE BOLTZMANN EQUATION

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Perusal of the Enskog-Chapman method appearing in its usual form in [1 and 2] or discussed in [3] shows clearly that the Enskog series for the distribution function satisfying the Boltzmann equation is asymptotic. This representation of the distribution function can also be deduced from the integral equation obtained by integrating the left and right-hand sides of the Boltzmann equation along the trajectory of the molecule (see e. g. [2]). Nevertheless, the asymptotic character of the Enskog expansion becomes particularly clear, if the integral form of the kinetic equation containing the probabilities of free paths of the molecules is used, and the Laplace's method applied in the asymptotic estimate of the integral.

We also see that the region of applicability of the resulting asymptotic representation is substantially curtailed (on the side of the high order of rarefaction) by discarding the corresponding exponential terms characterizing in particular the influence of the initial and boundary conditions.

1. We shall write the kinetic equation in the form

$$f(t, x_i, \xi_i) = f(t_0, x_{0i}, \xi_i) \exp\left(-\int_{t_0}^t N_\tau d\tau\right) + \int_{t_0}^t f^+(\tau, x_i - \xi_i(t-\tau), \xi_i) N_\tau \exp\left(-\int_\tau^t N_q dq\right) d\tau$$

$$N(t, x_i, \xi_i) = \int f(t, x_i, \xi_{i1}) \sigma g d\xi_1, \quad J^+ = \int f' f_1' g d\sigma d\xi_1$$

$$g = |\xi - \xi_1|, \quad N_\tau = N(\tau, x_i - \xi_i(t-\tau), \xi_i), \quad f^+ = J^- / N$$

Here  $N$  denotes the collision frequency,  $J^+$  is the integral of the reverse collisions,  $\sigma$  is the collision cross section and  $t - t_0$  is the time of the collisionless passage of a molecule with velocity  $\xi_i$  from some point  $x_{0i}$  to the point  $x_i = x_{0i} + \xi_i(t - \tau)$  under consideration.

Equations (1.1) show that the velocity distribution of the molecules at any instant  $t$  and at an arbitrary point  $x_i$ , is related to the values of the distribution function at any points of the region situated arbitrarily far from the point  $x_i$ , and at any instant of time preceding  $t$ . Nevertheless, the degree of mutual dependence of two points  $x_i$  and  $x_{0i}$  and of two instants  $t$  and  $t_0$ , decreases exponentially with increasing both the distance between these points and the time interval  $t - t_0$ , and the rate of decrease is directly related to the collision frequency. Our aim will be to find an explicit expression for the distribution function satisfying the kinetic equation, under the condition that the velocity distribution of the molecules at the point  $t, x_i$  is defined, basically, by the behavior of  $f$  in sufficiently